1 Boolean Algebra Definitions and Examples

Definition 1 *A **Boolean Algebra** is a tuple*

$$\langle \delta, \sqcap, \sqcup, ', \top, \bot \rangle$$

Where each element is defined as follows:

δ	*Non-empty set called the **domain***
$\sqcap$	*binary function* $\sqcap : \delta \times \delta \to \delta$
$\sqcup$	*binary function* $\sqcup : \delta \times \delta \to \delta$
$'$	*unary function* $' : \delta \to \delta$
$\bot$	*specific element called the **zero** element*
$\top$	*specific element called the **one** element*

Definition 2 *A Boolean algebra of sets* is any *Boolean algebra, where:*
δ is a set of sets,
$\sqcup$ is interpreted as set union, denoted $\cup$.
$\sqcap$ is interpreted as set intersection denoted $\cap$
$'$ is interpreted as set complement with repsect to $\top$, denoted , and $\sqsubseteq$ (or $\sqsupseteq$) is interpreted as set containtment, denoted as $\subseteq$ (or $\supseteq$).

Definition 3 *An* atom *of a Boolean algebra is an element $x \neq \bot$ such that there is no other element $y \neq \bot$ with $y \sqsubseteq x$. I can happen that there are no atoms at all in a Boolean algebra. In that case we call the Boolean algebra* atomless*; otherwise we call it* atomic*.*

Example 4

$$B_Z = \langle Powerset(\mathbb{Z}), \cap, \cup, ^-, \emptyset, \mathbb{Z} \rangle$$

*is a Boolean algebra of sets. In this algebra for each $i \in \mathbb{Z}$, the singleton $\{i\}$ is an atom. Thus this is an **atomic Boolean algebra**. Clearly $x = \{1\} \neq \bot$ and y can be either $\{1\}$ or $\emptyset$. We eliminate $\{1\}$ from consideration because it is not an **other element**. Since $y = \bot$, we conclude that there are no other elements such that $y \neq \bot$.*

Example 5 *Let H be the set of all finite unions of half-open intervals of the form $[a, b)$ over the rational numbers, where $[a, b)$ means all rational numbers that are greater than or equal to a and less than b, where a is a rational or $-\infty$ and b is a rational number or ∞.*

$$B_H = \langle H, \cap, \cup, ^-, \emptyset, \mathbb{Q} \rangle$$

This is another Boolean algebra of sets, but this set is atomless. $\forall x = [a, b)$, $a < b$ there exists a c such that $a < c < b$. Thus $[a, c) \subseteq [a, b)$, and $[a, c) \neq \emptyset$.